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14. ABSTRACT
There are a variety of systems whose properties are governed by the leading eigenvectors of the underlying system matrix. If the set of the dominant eigenvalues are clearly separated from the next largest eigenvalues, then via the corresponding leading eigenvectors, we can readily characterize the system properties. In this presentation, we will describe approach which can assure clear separation in leading eigenvalues by imposing a proper structure of the underlying matrix. Specifically, we provide bounds on eigenvalues for the hierarchical system connection structure. Based on these results, we can design hierarchical systems with assured clustering behavior or absence of it.

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A Multi-Hub Theory for Spectral Based System Design

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SIAM Conference on Discrete Mathematics

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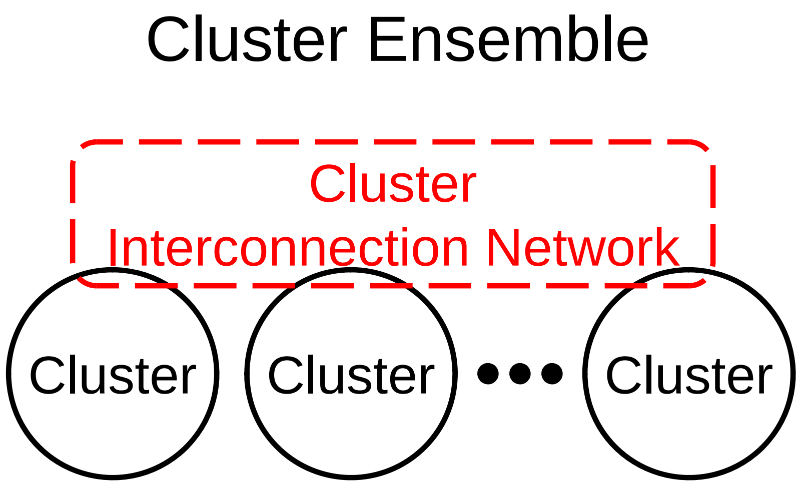
Austin, TX, USA

Background and Motivations

- Spectral analysis is a popular mathematical tool in analyzing network and distributed systems
- Given such a system (e.g., a sensor-target measurement system) it's often sufficient to examine only those **dominating** eigenvectors for which the corresponding eigenvalues are much larger than the rest of eigenvalues
- That is, these dominating eigenvectors can reveal important properties/states of the underlying system (e.g., detecting a malfunctioning sensor)

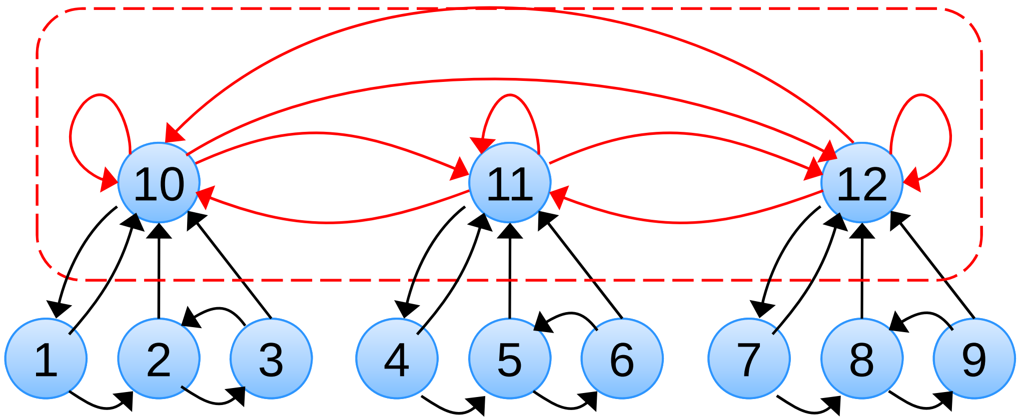
Objective of This Work

- We want to characterize graphs which exhibit strong dominance of the leading eigenvalues
- It is difficult to provide such characterization for general graphs, so we focus on some special graphs, which we call "cluster ensembles," as depicted to the right
- Cluster ensembles arise naturally when we construct a hierarchical system by interconnecting clusters



A Cluster Ensemble Example: Three Clusters Connected Via a Clique

Cluster Interconnection Network



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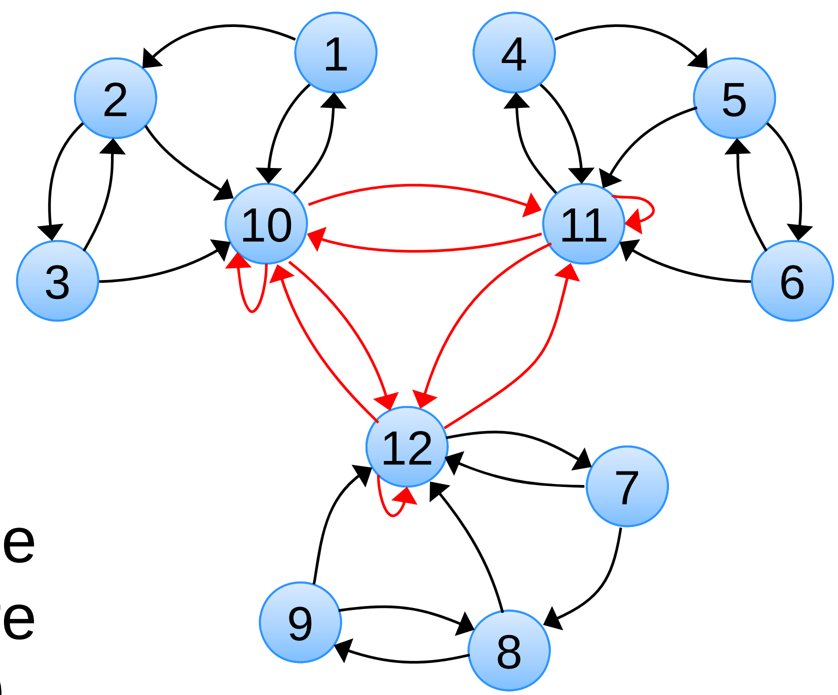


Cluster nodes which are part of the cluster interconnection network are called *hub nodes*. Thus, nodes 10, 11 and 12 are hub nodes

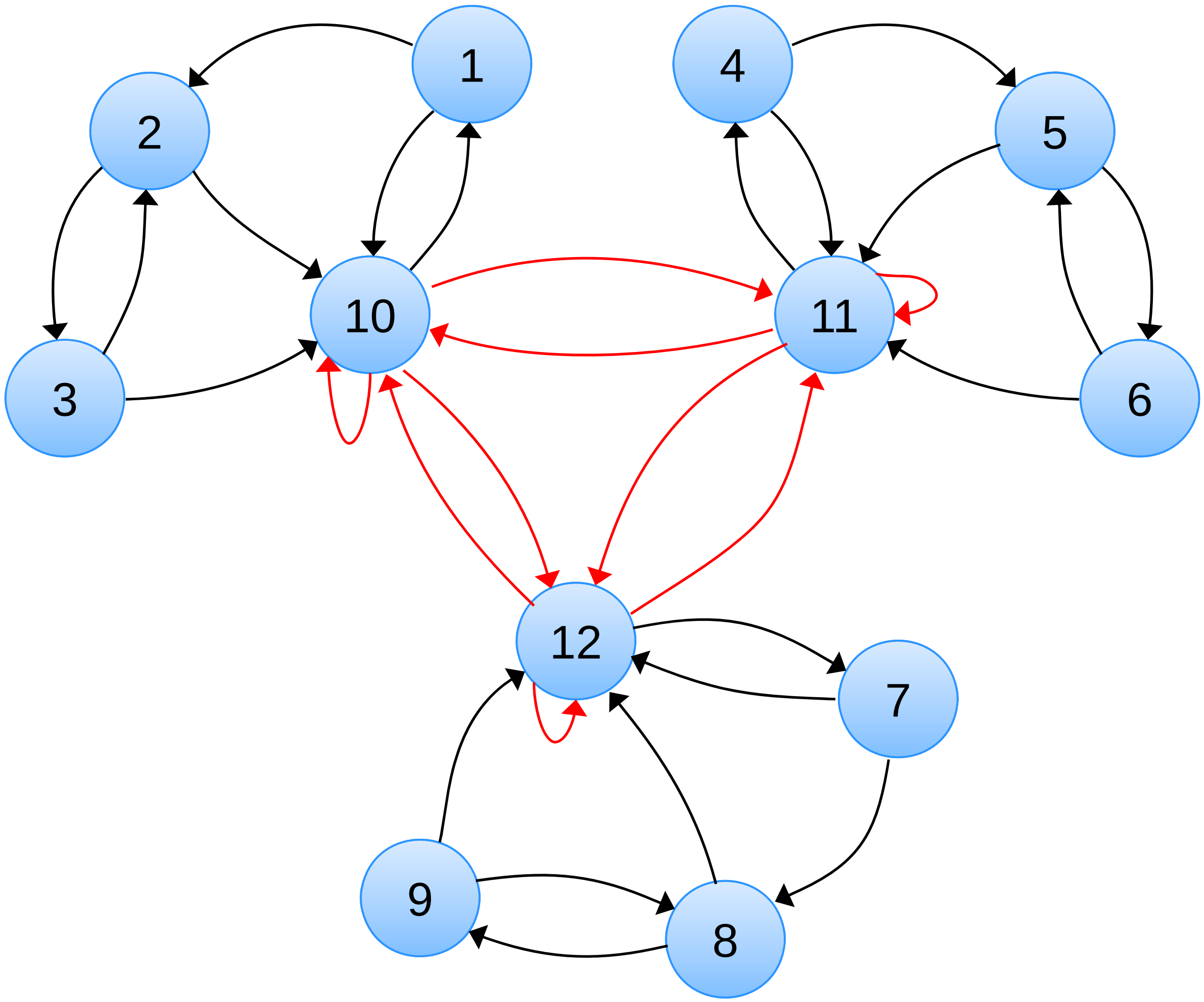
Furthermore, We May In Addition Assume Alpha-Beta Cluster Ensemble

This is a special cluster ensemble satisfying the following properties:

1. There are α identical clusters
2. Each cluster contains one hub node and β non-hub nodes. This means that the size of each cluster is β + 1.
3. In the adjacency matrix $\boldsymbol{C}$ of each cluster, columns of non-hub nodes are orthogonal to each other

It is possible to relax the above assumptions, e.g., for the orthogonality condition of item 3, $\boldsymbol{C}^T\boldsymbol{C}$ can go from a diagonal matrix to a diagonally dominant one

An Instance of an Alpha-beta Cluster Ensemble with α = 3 and β = 3



Notation for Eigenvalues

- In what follows, the eigenvalues of a symmetric matrix are always ordered such that

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$$

- When it is useful to indicate that they are the eigenvalues on a specific matrix ***F***, we write instead

$$\lambda_1(\boldsymbol{F}) \leq \lambda_2(\boldsymbol{F}) \leq \ldots \leq \lambda_n(\boldsymbol{F})$$

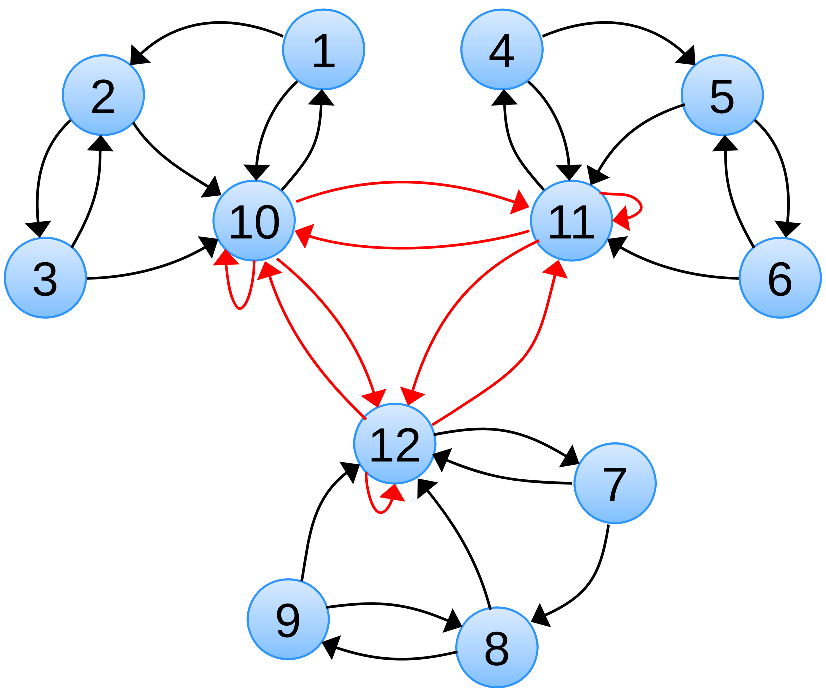
Preview of an Exemplar Result

- Consider an *alpha-beta cluster ensemble* of α identical clusters interconnected under a clique topology
- Each cluster contains β non-hub nodes
- Let m = α (β + 1) be the total number of nodes in the ensemble, and μ the largest in-degree of a non-hub node
- Let ***A*** be the adjacency matrix of the entire ensemble

Exemplar Theorem:

$$\frac{\lambda_{m-\alpha+1}(\boldsymbol{A}^T\boldsymbol{A})}{\lambda_{m-\alpha}(\boldsymbol{A}^T\boldsymbol{A})} \geq \frac{\beta}{\mu}$$

An instance of an alpha-beta cluster ensemble with α = 3 and β = 3

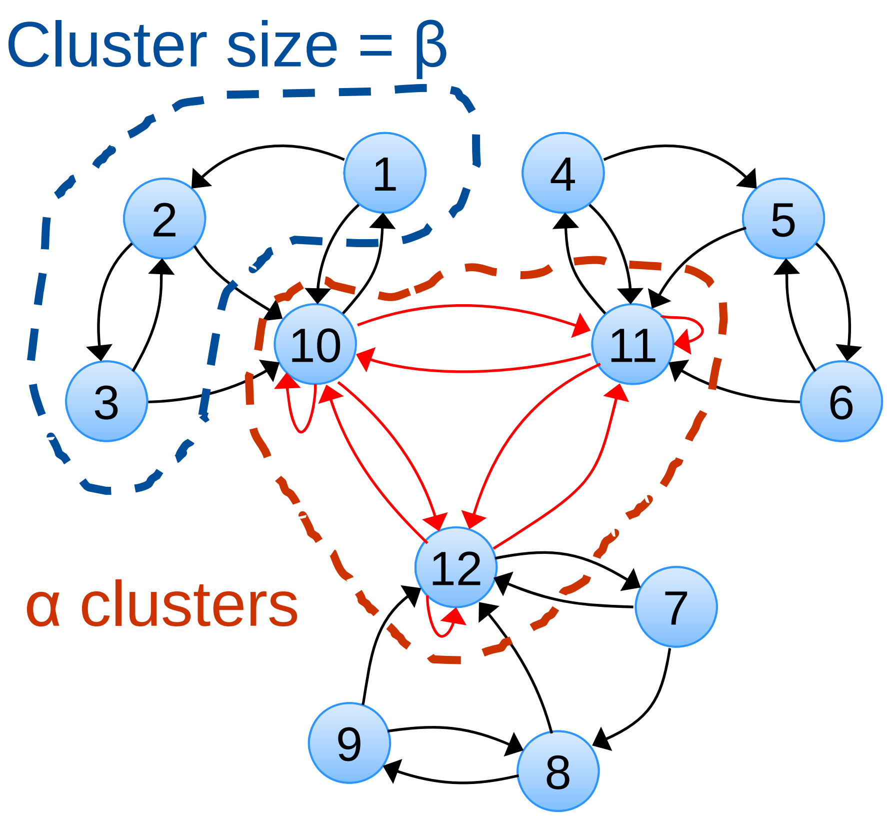


Implication of The Exemplar Theorem

Exemplar Theorem:

$$\frac{\lambda_{m-\alpha+1}(\boldsymbol{A}^T\boldsymbol{A})}{\lambda_{m-\alpha}(\boldsymbol{A}^T\boldsymbol{A})} \geq \frac{\beta}{\mu}$$

- As β increases, the separation ratio of the α leading eigenvalues from the rest of non-zero eigenvalues increase by a factor of at least β/μ. Thus when β is large, there are α dominating leading eigenvalues
- Thus we have succeeded in relating the dominance of the leading eigenvalues to cluster size β



Steps in Establishing Results of This Paper

1. Consider a general *m*-node, *k*-cluster ensemble
2. Partition the adjacency matrix ***A*** of the cluster ensemble and introduce the cluster interconnection submatrix ***B***
3. Derive bounds for the eigenvalues of $\boldsymbol{A}^T\boldsymbol{A}$ in terms of those of ***B***
4. Consider the special case of alpha-beta cluster ensembles, and derive bounds for the eigenvalues of ***B*** in terms of α and β
5. Derive bounds on the separation ratio for the α leading eigenvalues of an alpha-beta cluster ensemble, including bounds in the Exemplar Theorem described earlier

We will describe these steps in the rest of this presentation

Partitioning A^TA and Cluster Interconnection Submatrix

- For a general *m*-node, *k*-cluster ensemble, we can assume without loss of generality that its adjacency matrix ***A*** has the last *k* columns corresponding to the *k* hub nodes
- We express $\boldsymbol{A^TA}$ in a partitioned form:

$$\begin{array}{ccc} & \overbrace{\hphantom{T \qquad\quad X^T}}^{m} & \\ \boldsymbol{A^T A} = & \begin{pmatrix} \boldsymbol{T} & \boldsymbol{X} \\ \boldsymbol{X^T} & \boldsymbol{B} \end{pmatrix} & \begin{matrix} \\ k \end{matrix}\left.\vphantom{\begin{matrix} T \\ X^T \end{matrix}}\right\} m \\ & \hphantom{T \qquad\quad} k & \end{array}$$

- The ***B*** submatrix reflects the interconnections among the hub nodes and those between the hub nodes and other nodes in their respective clusters. We call ***B*** the *cluster interconnection submatrix*
- Our Theorem on the next slide shows that eigenvalues of ***B*** can contribute to upper and lower bounds for those of $\boldsymbol{A^TA}$

A Theorem Based on Cauchy's and Aronszajn's inequalities

$$\begin{array}{cc} & m \\ A^T A = & \overbrace{\begin{pmatrix} T & X \\ X^T & B \end{pmatrix}}^{} \left.\vphantom{\begin{pmatrix} T \\ X \end{pmatrix}}\right\} m \\ & k \end{array}$$

$$A^T A = \overbrace{\begin{pmatrix} T & X \\ X^T & B \end{pmatrix}_{k}}^{m} \Big\} m \quad \underset{k}{}$$

Theorem (Eigenvalue bounds for A^{T}A):

$$\lambda_j(A^T A) \le \lambda_j(T) \le \lambda_{k+j}(A^T A)$$

for $j = 1, \ldots, m-k$, and

$$\lambda_{k-j}(B) \le \lambda_{m-j}(A^T A) \le \lambda_{k-j}(B) + \lambda_{m-k}(T)$$

for $j = 0, \ldots, k-1$

Lower Bound on Eigenvalue Separation Ratio

$$\boldsymbol{A}^T\boldsymbol{A} = \begin{array}{cc} & \overbrace{\hphantom{\begin{matrix}\boldsymbol{T} & \boldsymbol{X}\end{matrix}}}^{m} \\ & \left.\begin{pmatrix} \boldsymbol{T} & \boldsymbol{X} \\ \boldsymbol{X}^T & \boldsymbol{B} \end{pmatrix}_{k}\right\} m \\ & \hphantom{\boldsymbol{T}\quad} k \end{array}$$

Corollary (Bound on eigenvalue separation ratio):

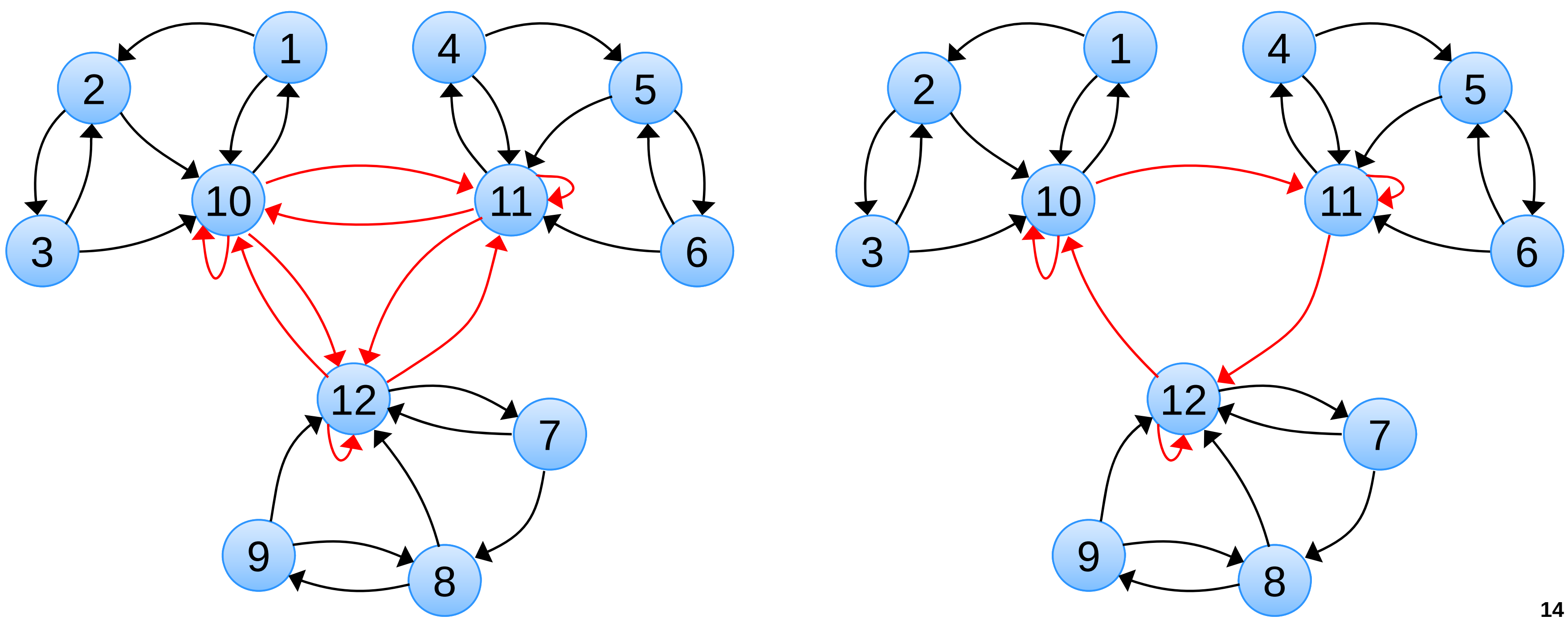
$$\frac{\lambda_{m-k+1}(\boldsymbol{A}^T\boldsymbol{A})}{\lambda_{m-k}(\boldsymbol{A}^T\boldsymbol{A})} \geq \frac{\lambda_1(\boldsymbol{B})}{\lambda_{m-k}(\boldsymbol{T})}$$

Cluster Interconnection Networks for Alpha-Beta Cluster Ensembles

We consider the two extreme topologies for the cluster interconnection network: *clique* (most dense) and *ring* (most sparse)

(a) Clique

(b) Ring



For the Case of Alpha-Beta Cluster Ensemble

Instead of:

$$\mathbf{A}^T \mathbf{A} = \begin{array}{c} \overbrace{\hphantom{\begin{pmatrix} \mathbf{T} & \quad \mathbf{X} \end{pmatrix}}}^{m} \\ \begin{pmatrix} \mathbf{T} & \mathbf{X} \\ \mathbf{X}^T & \mathbf{B} \end{pmatrix}_{k} \\ \qquad\qquad k \end{array} \left.\vphantom{\begin{pmatrix} \mathbf{T} \\ \mathbf{X}^T \end{pmatrix}}\right\} m$$

we have:

$$\mathbf{A}^T \mathbf{A} = \begin{array}{c} \overbrace{\hphantom{\begin{pmatrix} \mathbf{D} & \quad \mathbf{X} \end{pmatrix}}}^{m} \\ \begin{pmatrix} \mathbf{D} & \mathbf{X} \\ \mathbf{X}^T & \mathbf{B} \end{pmatrix}_{k} \\ \qquad\qquad k \end{array} \left.\vphantom{\begin{pmatrix} \mathbf{D} \\ \mathbf{X}^T \end{pmatrix}}\right\} m$$

where D is diagonal

Eigenvalues for the Cluster Interconnection Submatrix B

Consider the cluster interconnection network of an alpha-beta cluster ensemble

Theorem:

(a) If the hub interconnection is a clique, then

$$\lambda_j(\boldsymbol{B}) = \beta \text{ for } j = 1, \ldots, \alpha - 1$$
$$\lambda_\alpha(\boldsymbol{B}) = \alpha^2 + \beta$$

(b) If the hub interconnection is a ring, then

$$\beta \leq \lambda_j(\boldsymbol{B}) \leq \beta + 4 \text{ for } j = 1, \ldots, \alpha$$

Note: Theorem holds without the orthogonality assumption for non-hub columns in each cluster's adjacency matrix

Bounds on Eigenvalue Separation Ratios for $A^T A$

Consider an alpha-beta cluster ensemble. Suppose the cluster interconnection network is a clique or ring. We note: $m = \alpha\ (\beta + 1)$ and $k = \alpha$

Corollary:

(Exemplar Theorem)

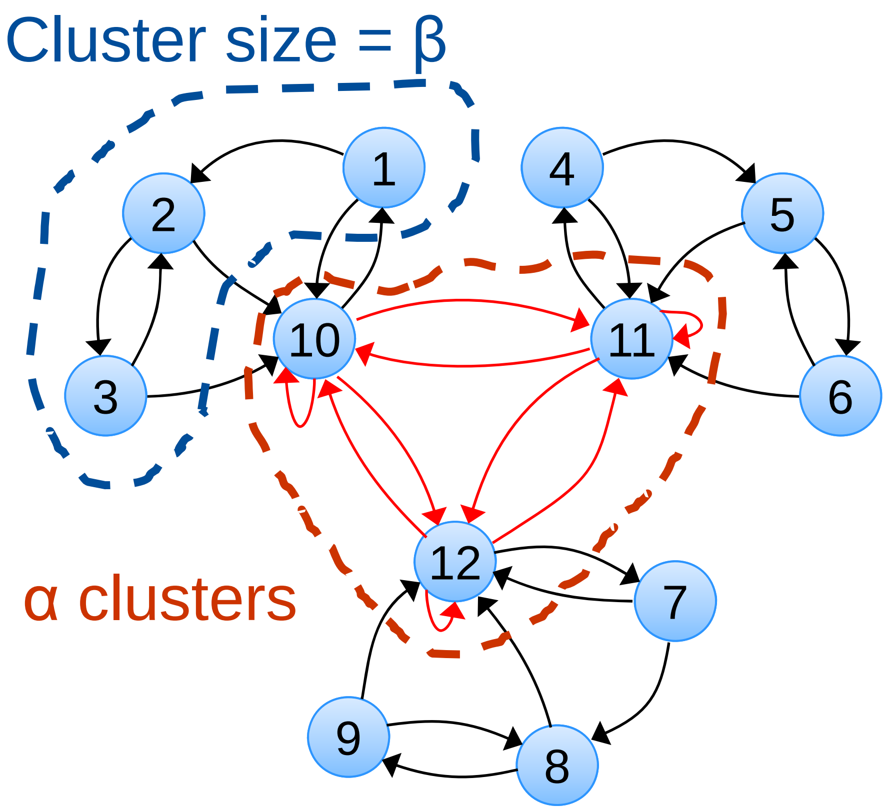
(a) $$\frac{\lambda_{m-k+1}(\boldsymbol{A}^T\boldsymbol{A})}{\lambda_{m-k}(\boldsymbol{A}^T\boldsymbol{A})} \geq \frac{\beta}{\mu}$$ (clique or ring)

(b) $$\frac{\lambda_{m}(\boldsymbol{A}^T\boldsymbol{A})}{\lambda_{m-k+1}(\boldsymbol{A}^T\boldsymbol{A})} \leq \frac{\beta + 4 + \mu}{\beta}$$ (ring)

(c) $$\frac{\lambda_{m}(\boldsymbol{A}^T\boldsymbol{A})}{\lambda_{m-1}(\boldsymbol{A}^T\boldsymbol{A})} \geq \frac{\alpha^2 + \beta}{\beta + \mu}$$ (clique)

System Implications

- When β increases, the separation ratio of the α leading eigenvalues and the rest of non-zero eigenvalues will increase by a factor of at least β/μ
- There are no significant separations among the α leading eigenvalues themselves
- This means the α leading eigenvectors will largely characterize the system



Summary and Conclusion

- We have motivated the problem of characterizing systems which exhibit strong dominance of the leading eigenvalues, in terms of underlying network topologies
- We have introduced a special class of networks called cluster ensembles, and established some fundamental mathematical results of ensuring such dominance
- As future work, we plan to generalize these characterization by allowing more general topologies than the alpha-beta ensemble and develop applications in **system design** based on spectral properties established in this paper